Malaysian Journal of Mathematical Sciences 16(4): 683–695 (2022)
https://doi.org/10.47836/mjms.16.4.03

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

The Convergence of Operator With Rapidly Decreasing Wavelet Functions

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Received: 12 March 2020
Accepted: 5 October 2022

Abstract

The expansion of (2D) wavelet functions with respect to $L^p(R^2)$ space converging almost everywhere for $1 < p < \infty$ throughout the length of the Lebesgue set points of space functions is investigated in this research. The convergence is established by assuming some wavelet function minimal regularity ψ_{j_1,j_2,k_1,k_2} under the current description of the wavelet projection operator known as 2D Hard Sampling Operator. Note that the feature of fast decline in 2D is derived here. Another condition is used, for instance, the wavelet expansion's boundedness under the Hard Sampling Operator. The bound (limit) is governed in magnitude with respect to the maximal equality of the Hardy-Littlewood maximal operator. Some ideas presented in this work to find a new method to prove the convergence theory for new type of conditional wavelet operator. Propose some conditions for wavelets functions and there expansion can support the operator to be convergence. It also perform a comparison with the identity convergent operator is our method for achieving this convergence.

Keywords: boundedness; convergence; maximal function; rapidly decreasing; wavelet expansion.

1 Introduction

The wavelet expansions almost everywhere convergence under a novel type of wavelet operator known as Hard Sampling Operator is investigated in this paper. For $1 < p < \infty$, these expansions are employed to further expand the $L^p(R^2)$ functions f in 2D. This is accomplished using wavelet functions that are quickly diminishing and bounded. Hard Sampling Operator $T_{\lambda_{1,2}}$ gives the f wavelet expansion as determined by the following Formula (1):

$$T_{\lambda_{1,2}} f(x,y) = \sum_{\mathbf{j} \geq \mathbf{J}; \{|\mathbf{a_{j,k}}| > \lambda_{\mathbf{1,2}}\}} \sum_{\mathbf{k}; \{|a_{\mathbf{j},\mathbf{k}}| > \lambda_{1,2}\}} a_{\mathbf{j},\mathbf{k}} \psi_{\mathbf{j},\mathbf{k}}, \tag{1}$$

where

$$a_{\mathbf{j},\mathbf{k}} = \int_{\mathbb{R}} \int_{\mathbb{R}} f(\chi, \zeta) \psi_{\mathbf{j},\mathbf{k}}(\chi, \zeta) d\chi d\zeta, \tag{2}$$

denotes a coefficient of expansion provided that

$$|a_{\mathbf{j},\mathbf{k}}| > \lambda_{1,2},$$

for $\mathbf{j} = \{(j_1, j_2) \in Z \times Z : j_1 = J_1, ..., \infty, j_2 = J_2, ..., \infty\}$, $\{\mathbf{k} = (k_1, k_2) \in Z \times Z\}$, $\lambda_{1,2}$ is a real number as well as $(x, y) \in R \times R$ denoting a Lebesgue point of f. Apart from that, the 2D wavelets basis functions $\{\psi_{\mathbf{j},\mathbf{k}}\} = \{2^{j_1/2} 2^{j_2/2} \psi(2^{j_1} \cdot - k_1) \psi(2^{j_2} \cdot - k_2)\}$ compose an orthonormal basis with respect to $L^2(R^2)$ under the effect of multi-resolution analysis.

The hard sampling operator convergence almost everywhere in L^p norm to f is demonstrated in this paper, which explores the almost everywhere convergent behavior of 2D wavelet expansion. In this study, any focus is limited to 2D wavelets, which analyses the quickness regarding the magnitude of wavelet expansions decrement, as well as the boundaries (limits) of wavelet expansions in 2D. In the following part, we will go through these features in depth.

2 Motivation and Innovative Results of the Work

In this work the problems on the convergence theory of wavelet expansions for the Hard Sampling Wavelet Operator in L^p are investigated for modelling scientific applications spaces with high-dimension. For improving the performance of Hard Sampling Wavelet Operator and employing wavelet basis functions which defined in the $\mathbb{R} \times \mathbb{R}$. Using the methods of multi-resolution analysis for wavelets expansions of the rapidly decreasing functions after determining that the multi-resolution analysis do not cause any loss of information during the analysis. This study presents expansion analysis of the wavelet basis functions to infinite-level of analysis. for more obvious whenever increases the number of partial sums' terms, the terms should be closer and closer to a certain function $f(x, y)$ (i.e. convergent). This work is completing the ideas of the work of the [11] by verifying two dimensional rapidly decreasing property for wavelet function which allows to achieve the almost everywhere convergence. Properties of the two-dimensional version of the Hard Sampling operator are obtained and applied to establish the proof of the almost everywhere convergence of the wavelet expansions of the rapidly decreasing functions. The bounds for the Hard Sampling operator is limited in its magnitude by the Hardy-Littlewood maximal operator. First main result of the work is given in the Theorem 4.1, where the convergence of the Hard Sampling operator at the points of the Lebesgue of the function being expanded. The important point in the paper is the inequality (7), where estimation for the two-dimensional wavelet basis functions are established. The results of the Theorem 2.3 are describing the behavior of the coefficients of the expansions of the rapidly decreasing function in terms wavelet expansions. The latter results can be obtained by standard methods of the multiresolution analysis after the estimation (7) is established. By reference to the results of the Theorem 4.2 and Theorem 4.3 the range for j_1 and j_2 in Hard Sampling operator are discovered in the Theorem 4.4. Finally using the obtained estimations the maximal operator of Hard Sampling operator is estimated and the almost every where convergence is derived from the latter estimation.

3 Related Works

Meyer [8] was the first to develop the wavelet expansion unconditional convergence in the multi-resolution analysis notion. Following that, numerous scholars investigated the subject of wavelet expansion convergence. For example, [10] modified non-separable Haar wavelet expansion as a wavelet transform to embed a binary watermark image into a color cover image. In another works, [4] and [3] used the radial decreasing as well as partial continuous wavelet functions to describe the convergence expansions of $L^p(R^n)$ functions with respect to $(1 \leqslant p < \infty)$ corresponding to every Lebesgue point of f. In a different study, [16] used regular orthogonal wavelets to expand the distribution, with the expansion converging pointwise to the distribution value. When a minimal regularity for ψ is taken into account, the pointwise convergence with respect to the wavelet projections operator may be achieved on the complete Lebesgue set of f [15]. Employing the approximation method, [17] converged the wavelet expansions of the $L^2(R)$ function to the mean value of both side limits at a generalized continuous point. Moreover, [2] employed the characterization of vector-valued Besov spaces function to examine the convergence property with respect to wavelet expansion that is non-divergence-free as well as divergence-free wavelet expansion. Furthermore, [5] used wavelet expansions to investigate the pointwise behavior of the Schwartz distributions of numerous variables. The study looked at the characterization of the quasi asymptotic behavior of distributions at finite points, as well as the relationships with measures $\alpha-$density points. In a different work, [13] used a prolate spheroidal wavelet to examine the pointwise convergence of wavelet expansions of $L^2(R)$ functions. In addition, [12]and [11] looked at how L^P functions defined on the S^2 and R^2 domains converge. By utilizing a spherical multi-resolution analysis on S^2 surface functions, [11] elucidated the pointwise behavior of spherical wavelet expansion with respect to the spherical wavelet projection operator. Hence, redefining the projection operator into 2D wavelet projections operators and expanding the 0-regular wavelet function with two scaling and shifting parameters, [12] enhanced Tao's work. To reach convergence, the effort entailed verifying the wavelet function's 2D fast decreasing property. In addition to what was mentioned above, a good approach to the topic of approximation was noticed in work of [7], Volterra integro-differential equation solved by applying Shannon approximation. On the other side, the Galerkin and the Petrov-Galerkin methods had been used to approximate the solution of nonlinear integral equation of the Urysohn type by work for [9]. After that, [1] produced another result about convergence theory by proving the strong uniform consistency properties of the non parametric linear wavelet-based estimators, over compact subsets of R^d, the corresponding rates of convergence were determinate. Liu [6] provided a fast convergent approximation to the nonlinear hyperbolic Schrödinger equations, the efficient method were presented precision by calculated the maximum error norm and the experimental rate of convergence.

4 The Convergent Behavior with Hard Sampling Operator

The findings of the work are presented in this section:

Theorem 4.1. *Let $\psi_{j,k}(x_1, x_2)$ is a 2-D wavelet basis function with 0-regularity, then for almost everywhere $\lambda_{1,2} \in R$, $(x, y) \in R \times R$ represents a Lebesgue point of $L^p(R^2)$ function $f(x, y)$, for $1 < p < \infty$, $\boldsymbol{j} = (j_1, j_2)$, $\boldsymbol{k} = (k_1, k_2)$ and $\boldsymbol{j}, \boldsymbol{k} \in Z \times Z$, which yields:*

$$\lim_{\lambda_{1,2} \to 0} T_{\lambda_{1,2}} f(x, y) = f(x, y). \tag{3}$$

Provided that the aforementioned circumstances are met, we now have

1.
$$|\psi_{\mathbf{j},\mathbf{k}}(x, y)| \leq \frac{\sigma_{j_1}\sigma_{j_2} c_N}{D(\mathbf{j}, \mathbf{k})^N}, \tag{4}$$
$$D(\mathbf{j}, \mathbf{k}) = \left[1 + \left|\left|2^{j_1}x - 2^{j_2}y\right| - |k_1 - k_2|\right|\right].$$

2.
$$|a_{\mathbf{j},\mathbf{k}}| \leq \frac{\sigma_{-j_1}\sigma_{-j_2} c_N M f(x, y)}{D(\mathbf{j}, \mathbf{k})^N}. \tag{5}$$

3.
$$\sup_{\lambda_{1,2}} \left|T_{\lambda_{1,2}} f(x, y)\right| \leq c_N M f(x, y) \tag{6}$$

where $\sigma_{j_1} = 2^{j_1/2}$, $\sigma_{j_2} = 2^{j_2/2}$, $\sigma_{-j_1} = 2^{-j_1/2}$ and $\sigma_{-j_2} = 2^{-j_2/2}$.
Proof: The next subsections explore and analyze all of the above conditions in order to establish this theorem.

4.1 Rapidly Decreasing Property

The given Theorem (4.2) derives the fast diminishing property of 2D wavelet function, thus developing the first requirement of Theorem (4.1):

Theorem 4.2. *Let $\psi_{\mathbf{j},\mathbf{k}}(x, y)$ is a 2-D wavelet basis function under 0-regularity with respect to $L^2(R^2)$ space in which $\boldsymbol{j} = (j_1, j_2)$, $\boldsymbol{k} = (k_1, k_2)$ as well as $\boldsymbol{j}, \boldsymbol{k} \in Z \times Z$. Therefore*

$$|\psi_{\mathbf{j},\mathbf{k}}(x, y)| \leqslant \frac{\sigma_{j_1}\sigma_{j_2} c_N}{[1 + |\alpha - \beta|]^N}, \tag{7}$$

$\sigma_{j_1} = 2^{j_1/2}$, $\sigma_{j_2} = 2^{j_2/2}$, $\alpha = \left|2^{j_1}x - 2^{j_2}y\right|$, $\beta = |k_1 - k_2|$ in which $N > 0$ as well as c_N represents constant.

Proof. Due to the fact that $\psi_{\mathbf{j},\mathbf{k}}(x, y)$ is specified in 2D, we now have

$$\psi_{\mathbf{j},\mathbf{k}}(x, y) = \psi_{j_1,k_1}(x)\psi_{j_2,k_2}(y)$$
$$= \sigma_{j_1}\sigma_{j_2}\psi(x_{j_1})\psi(y_{j_2}),$$

$x_{j_1} = 2^{j_1}x - k_1$, $y_{j_2} = 2^{j_2}y - k_2$. Because the set of wavelet basis functions $\{\psi_{j,k}\}(x) \in L^2(\mathbb{R}^n)$ is regarded as r-regular functions for $(r \in \mathbb{N})$, if ψ is chosen with the manners of:

$$|\psi_{j,k}(x)| \leq \frac{\sigma_j c_n}{[1 + |x_j|]^n}, \tag{8}$$

$\sigma_j = 2^{j/2}$, $x_j = 2^j x - k$. Therefore, $\psi_{\mathbf{j},\mathbf{k}}(x,y)$ resembles a 0-regular wavelet, and we may use the Inequality (8) given by:

$$|\psi_{j_1,k_1}(x)| \leqslant \frac{\sigma_{j_1} c_{N_1}}{[1 + |x_{j_1}|]^{N_1}},$$
$$|\psi_{j_2,k_2}(y)| \leqslant \frac{\sigma_{j_2} c_{N_2}}{[1 + |y_{j_2}|]^{N_2}},$$

with respect to $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^2$ and $N_1, N_2 > 0$. We now have

$$\begin{aligned} |\psi_{j_1,k_1}(x)\psi_{j_2,k_2}(y)| &\leq \frac{\sigma_{j_1}\sigma_{j_2} c_{N_1} c_{N_2}}{[1 + |x_{j_1}|]^{N_1} [1 + |y_{j_2}|]^{N_2}} \\ &\leq \frac{\sigma_{j_1}\sigma_{j_2} c_N}{[1 + |x_{j_1}| + |y_{j_2}| + |x_{j_1}|\,|y_{j_2}|]^N}, \end{aligned} \tag{9}$$

in which $N = min\{N_1, N_2\}$.
To elaborate on this, we see that,

$$\frac{1}{(1+t)^{N_1}} \times \frac{1}{(1+z)^{N_2}} \leqslant \begin{cases} \frac{1}{[(1+t)(1+z)]^{N_1}} & N_1 < N_2 \\ \frac{1}{[(1+t)(1+z)]^{N_2}} & N_2 < N_1. \end{cases}$$

Thus,

$$\frac{1}{(1+t)^{N_1}} \times \frac{1}{(1+z)^{N_2}} \leqslant \frac{1}{[(1+t)(1+z)]^N},$$

in which $N = min\{N_1, N_2\}$ as well as for all $t, z \in \mathbb{Z}^+$.
Suppose by taking the term

$$\begin{aligned} &\left[1 + \left|2^{j_1}x - k_1\right| + \left|2^{j_2}y - k_2\right| + \left|2^{j_1}x - k_1\right|\left|2^{j_2}y - k_2\right|\right] \\ &\geqslant 1 + \left|2^{j_1}x - k_1 + k_2 - 2^{j_2}y\right| + \left|2^{j_1}x - k_1\right|\left|2^{j_2}y - k_2\right| \\ &= 1 + \left|\left(2^{j_1}x - 2^{j_2}y\right) - (k_1 - k_2)\right| + \left|2^{j_1}x - k_1\right|\left|2^{j_2}y - k_2\right| \\ &\geqslant 1 + \left|\left|2^{j_1}x - 2^{j_2}y\right| - |k_1 - k_2|\right|. \end{aligned}$$

Therefore, formula (9) becomes

$$|\psi_{\mathbf{j},\mathbf{k}}(x,y)| = |\psi_{j_1,k_1}(x)\psi_{j_2,k_2}(y)| \leqslant \frac{\sigma_{j_1}\sigma_{j_2} c_N}{[1 + |\alpha - \beta|]^N}.$$

■ □

This completes the proof of Theorem (4.2).

Let us consider

$$D(\mathbf{j}, \mathbf{k}) = [1 + |\alpha - \beta|].$$

We obtain

$$|\psi_{\mathbf{j},\mathbf{k}}(x,y)| \leq \frac{\sigma_{j_1}\sigma_{j_2} c_N}{D(\mathbf{j}, \mathbf{k})^N}.$$

4.2 Boundedness of Wavelet Expansion

If the following Theorem (4.3) is proven to prove the wavelet expansion boundedness with respect to a Hard Sampling Operator, the second condition of Theorem (4.1) is true.

Theorem 4.3. *Let $\psi_{j,k}(x_1, x_2)$ denotes a 2-D wavelet basis function with 0-regularity with respect to wavelet expansion, then for almost everywhere $(x, y) \in R \times R$ represents a Lebesgue point of $f(x, y) \in L^p(R^2)$, for $1 < p < \infty$, $\boldsymbol{j} = (j_1, j_2)$, $\boldsymbol{k} = (k_1, k_2)$, $\boldsymbol{j}, \boldsymbol{k} \in Z \times Z$, $N > 0$, c_N denotes a constant whereas M resembles a maximal function operator of $f(x, y)$. Therefore, for $\sigma_{-j_1} = 2^{-j_1/2}$ and $\sigma_{-j_2} = 2^{-j_2/2}$*

$$|a_{\mathbf{j},\mathbf{k}}| \leq \sigma_{j_1}\sigma_{j_2}c_N \frac{Mf(x_1, x_2)}{D(\mathbf{j}, \mathbf{k})^N}.$$

Proof. Based on the Equation (2)

$$|a_{\mathbf{j},\mathbf{k}}| \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |\psi_{\mathbf{j},\mathbf{k}}(x, y)|\,|f(x, y)|\,dxdy.$$

We can implement the Dyadic interval requirement using the compact supported property for the wavelet function, which is specified by

$$I_{j,k} = [2^{-j}k, 2^{-j}(k + 1)), \tag{10}$$

with respect to each $j, k \in \mathbb{Z}$. A general index set for the wavelet basis functions is $\{I_{j,k}\}$, which is made up of these sub intervals collection. Applying Equation(7) in Theorem 4.2 yields

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\psi_{\mathbf{j},\mathbf{k}}(x, y)|dxdy = \int_{2^{-j_1}k_1}^{2^{-j_1}(k_1+1)}\int_{2^{-j_2}k_2}^{2^{-j_2}(k_2+1)} \frac{\sigma_{j_1}\sigma_{j_2}c_{N_1}c_{N_2}}{[1 + |\alpha - \beta|]^N} dxdy,$$

where $\sigma_{j_1} = 2^{j_1/2}$, $\sigma_{j_2} = 2^{j_2/2}$, $\alpha = \left|2^{j_1}x - 2^{j_2}y\right|$ and $\beta = |k_1 - k_2|$.
Suppose $D(\mathbf{j}, \mathbf{k}) = \text{int}\,[1 + |\alpha - \beta|]$.
Therefore, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\psi_{\mathbf{j},\mathbf{k}}(x, y)|dxdy = \int_{2^{-j_1}k_1}^{2^{-j_1}(k_1+1)}\int_{2^{-j_2}k_2}^{2^{-j_2}(k_2+1)} \frac{\sigma_{j_1}\sigma_{j_2}c_{N_1}c_{N_2}}{D(\mathbf{j}, \mathbf{k})^N} dxdy.$$

Hence,

$$|\psi_{\mathbf{j},\mathbf{k}}(x, y)| \leq \frac{\sigma_{j_1}\sigma_{j_2}c_N}{D(\mathbf{j}, \mathbf{k})^N}.$$

Thus,

$$|a_{\mathbf{j},\mathbf{k}}| \leq \frac{\sigma_{j_1}\sigma_{j_2}c_N \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x, y)|dxdy}{D(\mathbf{j}, \mathbf{k})^N},$$

$$|a_{\mathbf{j},\mathbf{k}}| \leq \frac{\sigma_{j_1}\sigma_{j_2}c_N Mf(x, y)}{D(\mathbf{j}, \mathbf{k})^N},$$

in which $c_N = c_{N_1}c_{N_2}$.
The proof has been completed ■ □

To locate the finest values for j_1 and j_2 in which the significant terms in the $T_{\lambda_{1,2}}$ summation may be shown, we then apply this to Theorem (4.4).

Theorem 4.4. *Let $\psi_{\boldsymbol{j},\boldsymbol{k}}(x_1, x_2)$ denotes a wavelet basis function under 0-regularity with respect to wavelet expansion $\sum_{\mathbf{j}}\sum_{\mathbf{k}} a_{\mathbf{j},\mathbf{k}}$, then the two scales j_1 and j_2 of Hard Sampling Operator in Equation (1) are varied by:*

$$J_1 \leq j_1 < J_1 + \aleph_1,$$
$$J_2 \leq j_2 < J_2 + \aleph_2,$$

where $\aleph_1 = c_{N_3} - 2N_1\log_2 D(j_1, k_1)$, $\aleph_2 = c_{N_4} - 2N_2\log_2 D(j_2, k_2)$ with having

$$|a_{\mathbf{j},\mathbf{k}}| > \lambda_{1,2}, \tag{11}$$

for $\lambda_{1,2}$ is a real number , $(x, y) \in \mathbb{R}\times\mathbb{R}$ denotes a Lebesgue point of $L^p(R^2)$ function $f(x, y)$ for $1 < p < \infty$ almost everywhere, $\mathbf{j} = \{(j_1, j_2) : j_1 \geq J_1, j_2 \geq J_2\}$, $\boldsymbol{k} = (k_1, k_2)$ with $\boldsymbol{j}, \boldsymbol{k} \in \mathbb{Z}\times\mathbb{Z}$, $[J_1 + \aleph_1] \in \mathbb{Z}$, $[J_2 + \aleph_2] \in \mathbb{Z}$, c_{N_3}, c_{N_4} resemble constants and $N_1, N_2 > 0$.

Proof. We now employ Inequality (5) with 1D wavelet function $\psi_{j_1,k_1}(x)$ in Theorem 4.1, which yields

$$|a_{j_1,k_1}| \leqslant \frac{\sigma_{-j_1} c_{N_1} Mf(x)}{D(j_1, k_1)^{N_1}},$$

and by implementing it with another 1D wavelet function $\psi_{j_2,k_2}(y)$, we obtain

$$|a_{j_2,k_2}| \leqslant \frac{\sigma_{-j_2} c_{N_2} Mf(y)}{D(j_2, k_2)^{N_2}}.$$

We now take into account $\lambda_1 \approx 2^{-J_1/2} Mf(x)$ and $\lambda_2 \approx 2^{-J_2/2} Mf(y)$. We replace Inequality (5) into Inequality (11) with 1D after employing each of the Inequalities (5) and (11), we have

$$2^{-J_1/2} Mf(x) < \frac{\sigma_{-j_1} c_{N_1} Mf(x)}{D(j_1, k_1)^{N_1}},$$

as well as

$$2^{-J_2/2} Mf(y) < \frac{\sigma_{-j_2} c_{N_2} Mf(y)}{D(j_2, k_2)^{-N_2}}.$$

This results into

$$2^{-J_1/2} < \frac{\sigma_{-j_1} c_{N_1}}{D(j_1, k_1)^{N_1}}; \qquad (a)$$
$$2^{-J_2/2} < \frac{\sigma_{-j_2} c_{N_2}}{D(j_2, k_2)^{N_2}}. \qquad (b)$$

Using log_2 with respect to both sides of the terms (a) and (b), we obtain

$$\log_2 2^{-J_1/2} < log_2 \sigma_{-j_1} + \log_2 c_{N_1} - N_1 \log_2 D(j_1, k_1);$$
$$\Rightarrow \frac{-J_1}{2} < \frac{-j_1}{2} + \log_2 c_{N_1} - N_1 \log_2 D(j_1, k_1);$$
$$\Rightarrow j_1 < J_1 + c_{N_3} - 2N_1 \log_2 D(j_1, k_1),$$

and this is lead to

$$j_1 < J_1 + \aleph_1$$

as well as

$$\begin{aligned} \log_2 2^{-J_2/2} &< \log{}_2\sigma_{-j_2} + \log_2 c_{N_2} - N_2 \log_2 D(j_2, k_2); \\ \Rightarrow \frac{-J_2}{2} &< \frac{-j_2}{2} + \log_2 c_{N_2} - N_2 \log_2 D(j_2, k_2); \\ \Rightarrow j_2 &< J_2 + c_{N_4} - 2N_2 \log_2 D(j_2, k_2), \end{aligned}$$

that is lead to

$$j_2 < J_2 + \aleph_2$$

in which $c_{N_1}, c_{N_2}, c_{N_3}$ and c_{N_4} resemble constants. Therefore, the finest scales j_1 as well as j_2 are in the range given below

$$\begin{aligned} J_1 \leq j_1 < J_1 + \aleph_1. \\ J_2 \leq j_2 < J_2 + \aleph_2. \end{aligned}$$

■ □

4.3 The limitation of Hard Sampling Operator

To limit the operator $T_{\lambda_{1,2}}$, we show that the maximal function described in the Hardy-Littlewood maximal operator M may be used to limit its magnitude, such as

$$Mf(x,y) = \sup_{r_1, r_2 > 0} \left| A_{r_1, r_2}(f)(x,y) \right|,$$

in which A_{r_1,r_2} denotes a maximal function given by

$$A_{r_1,r_2}(f)(x,y) = \frac{1}{|B(x,y,r_1,r_2)|} \int\limits_{B(x,r_1)} \int\limits_{B(y,r_2)} f(y_1,y_2) dy_1 dy_2.$$

Hence, the B ball measure as follows

$$|B(x,y,r_1,r_2)| = \sqrt{x_1^2 + x_2^2} \leq 2r^2,$$

for $(x_1, x_2) \in R \times R$ and the point $(0,0)$ is a center. To obtain further information, we direct the readers to ([14]).

Remark 4.1. *Generally, it is possible to bound* $\left\| T_{\lambda_{1,2}} f(x,y) - f(x,y) \right\|_{L^p}$ *provided the regularities of* f *and wavelet functions are determined.*

$$\begin{aligned} \left\| T_{\lambda_{1,2}} f(x,y) - f(x,y) \right\|_{L^p} = \\ \left\| (I - T_{\lambda_{1,2}}) f(x,y) \right\|_{L^p}. \end{aligned}$$

There appears a constant c_N *in which* I *is an identity operator when both* f *and the wavelet function* $\psi_{\mathbf{j},\mathbf{k}}$ *possess* N *continuous derivatives*

$$\left\| (I - T_{\lambda_{1,2}}) f(x,y) \right\|_{L^p} \leq \frac{c_N}{2^{N\mathbf{j}}} \left\| f(x,y) \right\|_{L^p}.$$

As a result, we may utilize the aforementioned facts to prove the given theorem as the third condition of the Theorem (4.1):

Theorem 4.5. *Let $\psi_{j,k}(x,y)$ denotes a 0-regular 2-D wavelet basis function with respect to wavelet expansion, hence, the Hard Sampling Operator given by*

$$T_{\lambda_{1,2}}f(x,y) = \sum_{|a_{\mathbf{j},\mathbf{k}}|>\lambda_{1,2}} \sum_{\mathbf{j}} \sum_{\mathbf{k}} a_{\mathbf{j},\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}},$$

possesses

$$\sup_{\lambda_{1,2}} \left|T_{\lambda_{1,2}}f(x_1,x_2)\right| \le c_N Mf(x_1,x_2).$$

With respect to $\lambda_{1,2} \in R \times R$, $(x_1,x_2) \in R \times R$ denotes a Lebesgue point of $L^p(R^2)$ function $f(x_1,x_2)$, for $1 < p < \infty$, $\mathbf{j} = \{(j_1,j_2) : j_1 \ge J_1, j_2 \ge J_2\}$, $\boldsymbol{k} = (k_1,k_2)$, as well as $\boldsymbol{j},\boldsymbol{k} \in Z \times Z$.

Proof:

Assuming the convergent Wavelet Projection Operator in [12], we can consider that the Wavelet Projection Operator $P_{\mathbf{J}}$ is written in the following equation

$$P_{\mathbf{J}}f(x,y) = \sum_{\mathbf{j}<\mathbf{J}} \sum_{\mathbf{k}} a_{\mathbf{j},\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}}, \tag{12}$$

converges almost everywhere with respect to $(x,y) \in \mathbb{R}^2$ a Lebesgue point of $f(x,y)$ as well as $\mathbf{j} = \{(j_1,j_2) : j_1 < J_1, j_2 < J_2\}$, $\boldsymbol{k} = (k_1,k_2)$ with $\boldsymbol{j},\boldsymbol{k} \in Z \times Z$. We now separate the terms in Equation (1) as well as Equation (12) as written below:

$$T_{\lambda_{1,2}}f(x,y) =$$

$$\sum_{\mathbf{j}<\mathbf{J};\{|a_{\mathbf{j},\mathbf{k}}|>\lambda_{1,2}\}} \sum_{\mathbf{k};\{|a_{\mathbf{j},\mathbf{k}}|>\lambda_{1,2}\}} a_{\mathbf{j},\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}} + \sum_{\mathbf{j}\ge\mathbf{J};\{|a_{\mathbf{j},\mathbf{k}}|>\lambda_{1,2}\}} \sum_{\mathbf{k};\{|a_{\mathbf{j},\mathbf{k}}|>\lambda_{1,2}\}} a_{\mathbf{j},\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}} \tag{13}$$

$$P_{\mathbf{J}}f(x,y) =$$

$$\sum_{\mathbf{j}<\mathbf{J};\{|a_{\mathbf{j},\mathbf{k}}|\le\lambda_{1,2}\}} \sum_{\mathbf{k};\{|a_{\mathbf{j},\mathbf{k}}|\le\lambda_{1,2}\}} a_{\mathbf{j},\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}} + \sum_{\mathbf{j}<\mathbf{J};\{|a_{\mathbf{j},\mathbf{k}}|>\lambda_{1,2}\}} \sum_{\mathbf{k};\{|a_{\mathbf{j},\mathbf{k}}|>\lambda_{1,2}\}} a_{\mathbf{j},\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}}. \tag{14}$$

By taking the term

$$\left|T_{\lambda_{1,2}}f(x,y) - P_{\mathbf{J}}f(x,y)\right|,$$

and replacing Equations (13) as well as (14), it yields

$$\left|T_{\lambda_{1,2}}f(x,y) - P_{\mathbf{J}}f(x,y)\right| =$$

$$\left|\sum_{\mathbf{j}\geqslant\mathbf{J};\{|a_{\mathbf{j},\mathbf{k}}|>\lambda_{1,2}\}} \sum_{\mathbf{k};\{|a_{\mathbf{j},\mathbf{k}}|>\lambda_{1,2}\}} a_{\mathbf{j},\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}} - \sum_{\mathbf{j}<\mathbf{J};\{|a_{\mathbf{j},\mathbf{k}}|\leqslant\lambda_{1,2}\}} \sum_{\mathbf{k};\{|a_{\mathbf{j},\mathbf{k}}|\leqslant\lambda_{1,2}\}} a_{\mathbf{j},\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}}\right|$$

$$\le \sum_{\mathbf{j}\ge\mathbf{J};\{|a_{\mathbf{j},\mathbf{k}}|>\lambda_{1,2}\}} \sum_{\mathbf{k};\{|a_{\mathbf{j},\mathbf{k}}|>\lambda_{1,2}\}} |a_{\mathbf{j},\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}}| + \sum_{\mathbf{j}<\mathbf{J};\{|a_{\mathbf{j},\mathbf{k}}|\le\lambda_{1,2}\}} \sum_{\mathbf{k};\{|a_{\mathbf{j},\mathbf{k}}|\le\lambda_{1,2}\}} |a_{\mathbf{j},\mathbf{k}}|\,|\psi_{\mathbf{j},\mathbf{k}}|$$

where $|a_{\mathbf{j},\mathbf{k}}| > \lambda_{1,2}$, j_1 and j_2 values range as follows

$$J_1 \le j_1 < J_1 + \aleph_1.$$
$$J_2 \le j_2 < J_2 + \aleph_2.$$

This results follows as

$$\left|T_{\lambda_{1,2}}f(x,y) - P_{\mathbf{J}}f(x,y)\right| \le \sum_{\substack{J_1\le j_1<J_1+\aleph_1\\ J_2\le j_2<J_2+\aleph_2}} \sum_{\mathbf{k}} |a_{\mathbf{j},\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}}| + \sum_{\mathbf{j}<\mathbf{J}} \sum_{\mathbf{k}} \lambda_{1,2}\,|\psi_{\mathbf{j},\mathbf{k}}|.$$

Note that there exist an integer $D(j_1, k_1) = d_1$ as well as $D(j_2, k_2) = d_2$ for $d_1 \geq 1$, $d_2 \geq 1$ and $d_{1,2} = d_1 \times d_2$ for every j_1 and j_2, as well as for every integer k_1 and k_2, we can deduce from Equation (4) and Equation (5) that

$$|\psi_{\mathbf{j},\mathbf{k}}(x,y)| \leq \frac{\sigma_{j_1}\sigma_{j_2}c_N}{D(\mathbf{j},\mathbf{k})^N}.$$

$$|a_{\mathbf{j},\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}}| \leq \frac{c_N M f(x,y)}{D(\mathbf{j},\mathbf{k})^N}.$$

This now yields

$$\begin{aligned}&\left|T_{\lambda_{1,2}} f(x,y) - P_{\mathbf{J}} f(x,y)\right| \leqslant \\ &\sum_{\substack{J_1 \leq j_1 < J_1 + \aleph_1 \\ J_2 \leq j_2 < J_2 + \aleph_2}} \sum_{d_{1,2}=1}^{\infty} \frac{c_N M f(x,y)}{d_{1,2}^N} + \sum_{j_1 < J_1} \sum_{j_2 < J_2} \sum_{d_{1,2}=1}^{\infty} \frac{\sigma_{j_1}\sigma_{j_2}\lambda_{1,2} c_N}{d_{1,2}^N}.\end{aligned} \tag{15}$$

Several estimations are needed to further clarify the Formula (15) as follows:

First: *Calculate the term $\sum_{d_{1,2}=1}^{\infty} d_{1,2}^{-N}$, for $N > 1$.*

Assume $t = d_{1,2} = d_1 \times d_2$. Hence, by employing integral test to locate the behavior of series, we get

$$\begin{aligned}\int_1^{\infty} t^{-N} dt &= \lim_{v \to \infty} \int_1^{v} t^{-N} dt \\ &= \lim_{v \to \infty} \left((v^{-N+1})(-N+1)^{-1} - (-N+1)^{-1}\right) \\ &= (N-1)^{-1} < \infty,\end{aligned}$$

which converges with respect to $N > 1$.

Therefore,

$$\sup_{d_{1,2}} \left(\sum_{d_{1,2}=1}^{\infty} d_{1,2}^{-N} \right) = 1.$$

Second: *Estimate the term $\sum_{0 \leq j_1 \leq J_1} \sum_{0 \leq j_2 \leq J_2} 2^{j_1/2} 2^{j_2/2}$.*

The following description will help us estimate the series:
Let $2^{\frac{1}{2}} = q$, and $\sum_{j_1=0}^{J_1} 2^{j_1/2} = S = 1 + q + q^2 + \cdots + q^{J_1}$
$qS = q + q^2 + \cdots + q^{J_1+1}$,
$qS - S = q^{J_1+1} - 1$,
$S(q-1) = q^{J_1+1} - 1$,
$S(q-1)(q+1) = (q^{J_1+1} - 1)(q+1)$,
$S = q^{J_1+2} + q^{J_1+1} - q - 1$,
$\leqslant q^{J_1+2} + q^{J_1+2} = 2q^{J_1+2}$.

Hence, $\sum_{j_1=0}^{J_1} 2^{j_1/2} \leqslant 4 2^{J_1/2}$. Likewise, we acquire $\sum_{j_2=0}^{J_2} 2^{j_2/2} \leqslant 4 2^{J_2/2}$.

Third: *Calculate the terms* $\sum\limits_{J_1 \leqslant j_1 < J_1 + \aleph_1} 1$ *and* $\sum\limits_{J_2 \leqslant j_2 < J_2 + \aleph_2} 1$,
$\aleph_1 = c_{N_3} - 2N_1 \log_2 d_1$ *and* $\aleph_2 = c_{N_4} - 2N_2 \log_2 d_2$.
To calculate the series by integral test, we discover that

$$\sum_{J \leqslant j < J + c_N - 2N \log_2 d} 1 = \int\limits_{J}^{J + c_N - 2N \log_2 d} dj = c_N - 2N \log_2 d < \infty.$$

Note that

$$\sum_{j=J}^{J+c_N-2N\log d} 1 = 1_J + 1_{J+1} + 1_{J+2} + \ldots + 1_{J+c_N-2N\log d}$$

$$\sup_{j_1} \left(\sum_{J_1 \leqslant j_1 < J_1 + c_{N_3} - 2N_1 \log_2 d_1} 1 \right) = 1_{J_1 + c_{N_3} - 2N_1 \log_2 d_1} = 1,$$

and

$$\sup_{j_2} \left(\sum_{J_2 \leqslant j_2 < J_2 + c_{N_4} - 2N_2 \log_2 d_2} 1 \right) = 1_{J_2 + c_{N_4} - 2N_2 \log_2 d_2} = 1.$$

We now replace the $\lambda_{1,2}$ *value with the First, Second as well as Third estimations in Equation* (15), *which yields*

$$\begin{aligned}
&\sum_{\substack{J_1 \leq j_1 < J_1 + \aleph_1 \\ J_2 \leq j_2 < J_2 + \aleph_2}} \sum_{d_{1,2}=1}^{\infty} \frac{c_N M f(x,y)}{d_{1,2}^N} + \sum_{\substack{j_1 < J_1 \\ j_2 < J_2}} \sum_{d_{1,2}=1}^{\infty} \frac{\sigma_{j_1} \sigma_{j_2} \lambda_{1,2} c_N}{d_{1,2}^N} \\
&\leqslant \sum_{\substack{J_1 \leq j_1 < J_1 + \aleph_1 \\ J_2 \leq j_2 < J_2 + \aleph_2}} \sum_{d_{1,2}=1}^{\infty} \frac{c_N M f(x,y)}{d_{1,2}^N} \\
&\quad + \sum_{d_{1,2}=1}^{\infty} \frac{\sigma_{J_1} \sigma_{J_2} 2^4 \sigma_{-J_1} \sigma_{-J_2} c_N M f(x,y)}{d_{1,2}^N} \\
&\leqslant \sum_{\substack{J_1 \leq j_1 < J_1 + \aleph_1 \\ J_2 \leq j_2 < J_2 + \aleph_2}} \sum_{d_{1,2}=1}^{\infty} \frac{c_N M f(x,y)}{d_{1,2}^N} + \sum_{d_{1,2}=1}^{\infty} \frac{c_N 2^4 M f(x,y)}{d_{1,2}^N}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sup \left| T_{\lambda_{1,2}} f(x,y) - P_{\mathbf{J}} f(x,y) \right| = \\
&\sup\Big(\sum_{\substack{J_1 \leq j_1 < J_1 + \aleph_1 \\ J_2 \leq j_2 < J_2 + \aleph_2}} \sum_{d_{1,2}=1}^{\infty} \frac{c_N M f(x,y)}{d_{1,2}^N} + \sum_{d_{1,2}=1}^{\infty} \frac{c_N c_M M f(x,y)}{d_{1,2}^N} \Big) \\
&= c_N M f(x,y) + c_{MN} M f(x,y).
\end{aligned}$$

Therefore,

$$\sup_{\lambda_{1,2}} \left| T_{\lambda_{1,2}} f(x,y) \right| \leq c M f(x,y),$$

where $c = c_N + c_{MN}$ *and* c, c_N, c_M, c_{MN} *are constants.* ■

5 Conclusion

This paper establishes the convergence of wavelet expansions for $L^p(R^2)$ functions. By validating several properties, the convergence was studied. The conclusion is outlined as follows:

- By validating the fast diminishing property of a 2D wavelet function, the boundaries of the wavelet function can be determined;
- The 2D wavelet expansion bounds under Hard Sampling Operator; as well as
- The Hard Sampling Operator bounds (limits) is obtained by implementing maximal inequality of Hardy-Littlewood maximal operator.

The results of this research may be used to establish the convergence of 2D Hard Sampling Operators practically anywhere using strategies such as

- Applying multi-resolution analysis;
- Employing holder equality;
- The identity operator bounded property;
- The Hardy-Littlewood maximal operator's bounded condition.

Acknowledgement We are very grateful to anonymous reviewers for their comments and constructive suggestions that facilitated us in improving this article.

Conflicts of Interest The authors declare that have no conflict of interest. and have no any personal circumstances or interest that may be perceived as inappropriately influencing the representation or interpretation of reported research results.

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